QUESTION BANK

REAL ANALYSIS

SEMESTER-2

Q 1 Let a, b, c be any elements of $\mathbf{R}$. Show that :

(*i*) If $a > b$ and $b > c$, then $a > c$.

(*ii*) If $a > b$, then $a + c > b + c$.

(*iii*) If $a > b$ and $c > 0$, then $ca > cb$.

Q 2 Find all $x \in \mathbf{R}$ that satisfy the inequality :

$$4 < |x + 2| + |x - 1| < 5.$$

Q 3 If $y > 0$, show that there exists $n \in \mathbf{N}$ such that $1/2^n < y$. Justify each step by referring to an appropriate property or theorem.

Q 4 State the Completeness Property of $\mathbf{R}$. Show that if A and B are bounded subsets of $\mathbf{R}$, then :

$$\sup(A \cup B) = \sup\{\sup A, \sup B\}.$$

Q 5 Show that intersection of any arbitrary collection of closed sets in **R** is closed. Show, by an example, that union of infinitely many closed sets in **R** used not be closed.

Q 6 Prove that a sequence in **R** can have at most one limit.

Q 7 Suppose every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.

Q 8 Show that the sequence $(a_n) = ((-1)^n)$ does not converge.

Q 9 Let (s_n) be a sequence in **R**. Prove that $\lim (s_n) = 0$ if and only if $\lim (|s_n|) = 0$.

Q 10 Let (s_n) be a sequence that converges. Show that if $s_n \geq a$ for all but finitely many n, then $\lim (s_n) \geq a$.

Q 11 Let $X = (x_n)$ be a bounded increasing sequence.

Show that X is convergent and :

$$\lim (x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

Q 12 Show that $\lim \left(\sqrt{n} + 7\right) = +\infty$.

Q 13 Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Show that the sequence $\left(\sqrt{x_n}\right)$ of positive square roots converges and $\lim \left(\sqrt{x_n}\right) = \sqrt{x}$.

Q 14 Let (s_n) and (t_n) be the following sequences that repeat in cycles of four :

$$(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots\ldots\ldots)$$

$$(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \ldots\ldots\ldots)$$

Find $\lim\inf (s_n + t_n)$ and $\lim\sup (s_n + t_n)$.

Q 15 Define a Cauchy sequence. Show that the sequence :

$$\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right)$$

is divergent.

Q 16 Using squeeze theorem or otherwise, determine the limit of the sequence $\left(n^{1/n^2}\right)$.

Q 17 State and prove the comparison test for series.

Q 18 Test the convergence of :

(i) $\sum \frac{1}{2^n + n}$

(ii) $\sum \frac{(-1)^n n!}{2^n}$.

Q 19 Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges. Also, give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges. Justify your answers.

Q 20 State and prove the Alternating Series Theorem.

Q 21 Test the convergence of :

(*i*) $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)}$

(*ii*) $\sum \frac{(-1)^n}{n}$.

Q 22 Show that if $\sum a_n$ converges, then $\lim a_n = 0$. Show, by an example, that the converse is not true.

Q 23 Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbf{N}$. Show that (x_n) is bounded and monotone. Find its limit.